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LETTER TO THE EDITOR

Basic representations of $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ and the polynomial solutions to the reduced BKP hierarchies

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Abstract. Basic representations of $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ are studied. The weight vectors are represented in terms of Schur's Q -functions. The method to get the polynomial solutions to the reduced BKP hierarchies is shown to be equivalent to a certain rule in the Maya game.

Nonlinear integrable systems are important in mathematical physics. Among others the KP hierarchy has been deeply studied in various aspects such as representation theory, algebraic geometry and two dimensional gravity. In 1981 Date *et al* introduced a KP-like hierarchy of nonlinear differential equations, which has a symmetry of the infinite dimensional Lie algebra $B_\infty = \mathfrak{go}(\infty)$ and called it the KP hierarchy of B type or BKP hierarchy for short [1, 2]. Although this hierarchy has nice properties which reflect the Lie algebra symmetry, it seems that little research has been made compared with the KP hierarchy. This letter is an attempt to give an explicit expression of the weighted homogeneous solutions to the reduced BKP hierarchies.

We first review some ingredients of the BKP hierarchy and affine Lie algebras for the following discussions.

Let $\phi_n (n \in \mathbb{Z})$ be the neutral free fermion satisfying

$$\{\phi_m, \phi_n\} = (-1)^m \delta_{m+n,0}. \tag{1}$$

Note that $\phi_0^2 = 1/2$. We denote by $B = B_0 \oplus B_1$ the superalgebra generated by the neutral free fermions. Let $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = B_0|0\rangle \oplus B_1|0\rangle$ (resp. $\mathcal{F}^* = \langle 0|B_0 \oplus \langle 0|B_1$) be the Fock space (resp. dual Fock space), where the vacuum $|0\rangle$ (resp. $\langle 0|$) is defined by

$$\phi_n|0\rangle = 0 (n < 0) \quad (\text{resp. } \langle 0|\phi_n = 0 (n > 0)). \tag{2}$$

The vacuum expectation value $\langle 0|a|0\rangle (a \in B)$ is uniquely determined by setting $\langle 0|1|0\rangle = 1, \langle 0|\phi_0|0\rangle = 0$.

We construct the realization of \mathcal{F} . The normal ordering for the neutral fermions is defined by

$$:\phi_i\phi_j := \phi_i\phi_j - \langle 0|\phi_i\phi_j|0\rangle. \tag{3}$$

Define the Hamiltonian by

$$H(t) = \frac{1}{2} \sum_{j \geq 1, \text{ odd } n \in \mathbb{Z}} (-1)^{n+1} t_j \phi_n \phi_{-j-n}. \tag{4}$$

Let $V = \mathbb{C}[t_1, t_3, t_5, \dots]$ be a polynomial ring. There is an isomorphism between \mathcal{F}_0 and V , defined by

$$a|0\rangle \rightarrow \langle 0|e^{H(t)}a|0\rangle \quad a \in \mathcal{B}_0. \tag{5}$$

The subspace

$$B_\infty = \left\{ \sum_{m,n \in \mathbb{Z}} c_{mn} : \phi_m \phi_n ; c_{mn} = 0 \text{ if } |m - n| \gg 0 \right\} \tag{6}$$

of \mathcal{B}_0 admits a structure of a Lie algebra isomorphic to the one dimensional central extension of $\mathfrak{o}(\infty)$. The representation of B_∞ on V is defined by the vertex operator

$$Z(p, q) = \frac{1}{2} \frac{1 - q/p}{1 + q/p} \left\{ \exp \left(\sum_{j: \text{ odd}} (p^j - q^j) t_j \right) \exp \left(-2 \sum_{j: \text{ odd}} \frac{1}{j} (p^{-j} - q^{-j}) \frac{\partial}{\partial t_j} \right) - 1 \right\} \tag{7}$$

which corresponds to the action of $:\phi(p)\phi(q):$ on \mathcal{F}_0 , where $\phi(p) = \sum_{n \in \mathbb{Z}} \phi_n p^n$. The space of the solutions (τ -functions) to the BKP hierarchy is obtained as the group orbit through $1 \in V$ under this representation.

We consider the reduction of the BKP hierarchy [3]. For a fixed positive integer r , r -reduced BKP hierarchy is defined through the specialization of the vertex operator (7)

$$q = p\omega \quad \omega^r = 1 \quad \omega \neq 1. \tag{8}$$

This specialization is known to be equivalent to restricting the algebra B_∞ to the subalgebra $A_{2l}^{(2)}$ for $r = 2l + 1$, and $D_{l+1}^{(2)}$ for $r = 2l + 2$. The representation space will be denoted by V_r . It should be noted that the situation is different between odd r and even r . In the $r = 2l + 1$ case the specialization means the deletion of the variables $t_j (j \equiv 0 \pmod r)$. Hence the representation space becomes smaller: $V_r = \mathbb{C}[t_j; j \geq 1, \text{ odd and } j \not\equiv 0 \pmod r]$. On the other hand, in the $r = 2l + 2$ case, since the representation space has no variables with an even index, the space itself is unchanged: $V_r = V$.

In order to compute the weight of a given vector, we write down coroots of $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ explicitly by means of the neutral fermion operators:

$$A_{2l}^{(2)} \quad (r = 2l + 1)$$

$$\begin{aligned} \alpha_0^\vee &= 2 \sum_{n \in \mathbb{Z}} (-1)^{n+1} : \phi_{nr-1} \phi_{-nr+1} : + 1 \\ \alpha_i^\vee &= \sum_{n \in \mathbb{Z}} (-1)^{n+i} (: \phi_{nr+i} \phi_{-nr-i} : - : \phi_{nr-(i+1)} \phi_{-nr+(i+1)} :) \quad (1 \leq i \leq l-1) \\ \alpha_l^\vee &= \sum_{n \in \mathbb{Z}} (-1)^{n+l} : \phi_{nr+l} \phi_{-nr-l} : . \end{aligned}$$

$$D_{l+1}^{(2)} \quad (r = 2l + 2)$$

$$\alpha_0^\vee = -2 \sum_{n \in \mathbb{Z}} : \phi_{nr-1} \phi_{-nr+1} : + 1$$

$$\alpha_i^\vee = \sum_{n \in \mathbb{Z}} (-1)^i (: \phi_{nr+i} \phi_{-nr-i} : - : \phi_{nr-(i+1)} \phi_{-nr+(i+1)} :) \quad (1 \leq i \leq l-1)$$

$$\alpha_l^\vee = 2 \sum_{n \in \mathbb{Z}} (-1)^l : \phi_{nr+l} \phi_{-nr-l} : .$$

Let $\delta = \sum_{i=0}^l a_i \alpha_i$ be the fundamental imaginary root of the affine Lie algebra, where a_i are the labels of the corresponding generalized Cartan matrix. For $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ it reads $\delta = 2 \sum_{i=0}^{l-1} \alpha_i + \alpha_l$ and $\delta = \sum_{i=0}^l \alpha_i$, respectively.

Finally we recall the basic representation of affine Lie algebras. The irreducible representation with highest weight Λ_0 is called the basic representation, where

$$\Lambda_0(\alpha_0^\vee) = 1 \quad \Lambda_0(\alpha_j^\vee) = 0 (j \neq 0). \tag{9}$$

The representation on \mathcal{F}_0 constructed above turns out to be the basic representation $L(\Lambda_0)$ of $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ [6].

We now consider weight vectors, Q -functions and the Maya game. It is shown in [6] that a vector

$$(m_1, \dots, m_{2d}) := \phi_{m_{2d}} \dots \phi_{m_1} |0\rangle \in \mathcal{F}_0 \quad (m_{2d} > \dots > m_1 \geq 0) \tag{10}$$

is a weight vector of the Fock representation of B_∞ . Since $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ are Lie subalgebras of B_∞ , the vector (10) is also a weight vector of $L(\Lambda_0)$ of these affine Lie algebras. In view of the principally specialized character of $L(\Lambda_0)$, we have the following proposition.

Proposition 1. The basic representation $L(\Lambda_0)$ of $A_{2l}^{(2)}$ (resp. $D_{l+1}^{(2)}$) has the basis consisting of the weight vectors

$$\{(m_1, \dots, m_{2d}); m_{2d} > \dots > m_1 \geq 0, m_j \not\equiv 0 \pmod{2l+1}\} \tag{11}$$

$$\text{(resp. } \{(m_1, \dots, m_{2d}); m_{2d} > \dots > m_1 \geq 0\}). \tag{12}$$

Let us turn to the realization V_r of the basic representation $L(\Lambda_0)$, where $r = 2l + 1$ for $A_{2l}^{(2)}$ and $r = 2l + 2$ for $D_{l+1}^{(2)}$. We see how each weight vector of $L(\Lambda_0)$ is expressed explicitly as a polynomial in V_r . To this end we recall Schur's Q -functions [8]. Let X_1, \dots, X_n be indeterminates. The Hall-Littlewood symmetric function indexed by the Young diagram $Y = (m_1, \dots, m_n) (m_n \geq \dots \geq m_1 \geq 0)$ is defined by

$$Q_Y(X_1, \dots, X_n; q) = \prod_{j \geq 1} (q; q)_{\mu_j} \sum_{w \in \mathfrak{S}_n / \mathfrak{S}_n^Y} w \left(X_1^{m_n} \dots X_n^{m_1} \prod_{m_i < m_j} \frac{X_i - qX_j}{X_i - X_j} \right) \tag{13}$$

where q is a parameter, $\mu_j = \#\{i; m_i = j\}$ is the multiplicity of j in Y , $(q; q)_k = \prod_{i=1}^k (1 - q^i)$ and \mathfrak{S}_n^Y is the subgroup consisting of permutation w which does not change the diagram Y , i.e., $m_{w(i)} = m_i$ for $1 \leq i \leq n$. As a specialization $q = -1$ we get Schur's

Q -function $Q_Y(X_1, \dots, X_n)$. It is obvious from (13) that $Q_Y(X_1, \dots, X_n) = 0$ unless Y is a strict Young diagram, i.e., $\mu_j = 1$ for any $j \geq 1$.

The Frobenius formula gives the relation between Q -functions and power sum symmetric functions $p_m(X_1, \dots, X_n) = X_1^m + \dots + X_n^m$:

$$Q_Y(X_1, \dots, X_n) = \sum_{Y'} 2^{l(Y')} z_{Y'}^{-1} \chi_{Y'}^Y(-1) \prod_{j=1}^n p_{m_j'}(X_1, \dots, X_n). \tag{14}$$

Here the summation runs over all Young diagrams $Y' = (m_1', \dots, m_n')$ consisting of odd numbers m_j' , $l(Y')$ is the number of non-zero m_j' 's, and $z_{Y'} = \prod_{j \geq 1} j^{v_j} v_j!$, where v_j denotes the multiplicity of j in Y' . Although we do not make explicit, $\chi_{Y'}^Y(-1)$ is an integer computed easily from the character table of projective representations of the symmetric group S_n . Introduce new variables $t_j = (2/j)p_j(X_1, \dots, X_n)$ for positive odd integers j , so that the Q -function turns out to be a weighted homogeneous polynomial of $t = (t_1, t_3, \dots)$:

$$Q_Y(t) = \sum_{Y'} \chi_{Y'}^Y(-1) \frac{t_1^{v_1} t_3^{v_3} \dots}{v_1! v_3! \dots} \in V. \tag{15}$$

Another expression is known for the Q -functions [5]. Let $Y = (m_1, \dots, m_{2d})(m_{2d} > \dots > m_1 \geq 0)$ be a strict Young diagram, and consider $2d \times 2d$ skew symmetric matrix $(Q_{(m_i, m_j)}(t))$. Then we have

$$Q_Y(t) = \text{Pf}(Q_{(m_i, m_j)}(t)) \tag{16}$$

where Pf denotes the Pfaffian. Using anti-commutation relations, it is easily deduced that

$$\langle 0 | e^{H(t)} \phi_m \phi_n | 0 \rangle = \frac{1}{2} Q_{(m, n)}(t) \quad (m > n \geq 0) \tag{17}$$

and

$$\langle 0 | e^{H(t)} \phi_{m_{2d}} \dots \phi_{m_1} | 0 \rangle = \frac{1}{2^d} \text{Pf}(Q_{(m_i, m_j)}(t)). \tag{18}$$

Hence every weight vector of the Fock representation of B_∞ is expressed by means of a single Q -function.

For a polynomial $P(t) \in V$, we denote by $P(t')$ the polynomial in V_r obtained from $P(t)$ by putting $t_{j_r} = 0$ for $j = 1, 2, \dots$. Our claim is as follows:

Proposition 2. The basic representation on V_r of $A_{2r}^{(2)}$ with $r = 2l + 1$ (resp. $D_{l+1}^{(2)}$ with $r = 2l + 2$) has the basis consisting of weight vectors

$$\{Q_Y(t'); Y = (m_1, \dots, m_{2d}), m_{2d} > \dots > m_1 \geq 0, m_j \not\equiv 0 \pmod{2l + 1}\} \tag{19}$$

$$\text{(resp. } \{Q_Y(t'); Y = (m_1, \dots, m_{2d}), m_{2d} > \dots > m_1 \geq 0\}). \tag{20}$$

The Maya game is a fermion version of the so called 'Nim' defined as follows [4]. Consider infinitely many cells indexed by Z . Each $m \in Z$ is said to be either black or white according to whether it is filled with a single particle or is empty. The boundary condition is that m (resp. $-m$) is white (resp. black) for sufficiently large $m > 0$. The vacuum is a state such that m is black for all $m < 0$ and is white for all $m \geq 0$ (figure 1). In the general state

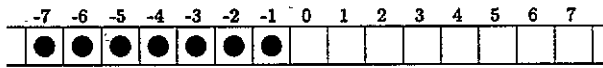


Figure 1. Maya diagram of the vacuum state.

some particles excite and occupy some non-negative numbered cells. The game, played by two, is to move a particle in turn to a lower level until one completes the vacuum. A player loses the game if he faces stalemate, i.e., the vacuum state.

To each weight vector (m_1, \dots, m_{2d}) we associate a state of the Maya game in the following manner. Every positive cell m_j is filled with a particle excited from the cell $-m_j$. There is an ambiguity at the cell indexed 0. We choose the convention that 0 is always white. The state of the Maya game such obtained from a weight vector is called a symmetric state.

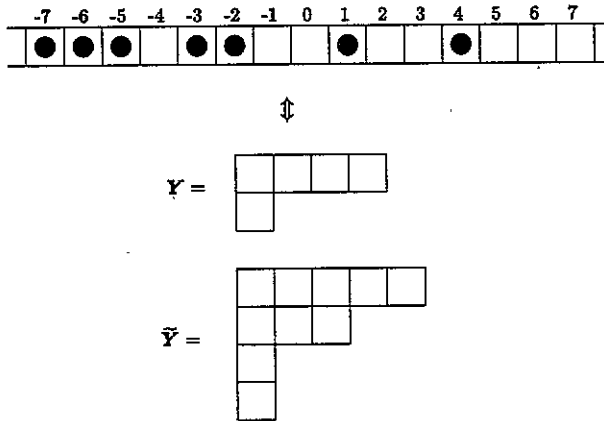


Figure 2. The symmetric state for $(1, 4)$ and the corresponding Young diagrams.

In the language of Young diagrams the relation between symmetric states and weight vectors is described as follows. For a weight vector (m_1, \dots, m_{2d}) , draw the corresponding strict Young diagram Y . Let \tilde{Y} be the shift symmetric Young diagram obtained from Y , which is explained in [8]. Then the rule that associates a state of Maya game with a Young diagram leads to the symmetric state corresponding to (m_1, \dots, m_{2d}) (cf [4]) (figure 2). We remark that the transposed Young diagram ${}^t\tilde{Y}$ leads to the symmetric state with black 0.

We now state how to obtain weight vectors of weight $\lambda \pm \delta$ from a given weight vector of weight λ . The procedure is described by modifying the rule of the Maya game.

First we consider the case of $A_{2l}^{(2)}$. Suppose a weight vector of weight λ is given. Then one of the following possible moves is permitted to have weight $\lambda - \delta$.

- (1) Move a black by $2l + 1$ to the right.
- (2) Replace the pair of white cells $(k, 2l + 1 - k)$ by black for $1 \leq k \leq l$.

For the case $D_{l+1}^{(2)}$ the moves are:

- (1) Replace the white $l + 1$ by black.
- (2) Move a black at a positive multiple of $l + 1$ to the right.
- (3) Once the cell $n(2l + 2)$ becomes black for an $n \geq 1$, instead of that single black, the black pairs $(k, n(2l + 2) - k)$ are also permitted for $1 \leq k < n(2l + 2)$.

The inverse moves give weight vectors of weight $\lambda + \delta$ from one of weight λ .

Next we characterize the maximal weight vectors of the basic representation $L(\Lambda_0)$ of the affine Lie algebras $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ and, as a consequence, obtain the weighted homogeneous polynomial solutions to the corresponding reduced BKP hierarchy.

By definition the weight λ of $L(\Lambda_0)$ of $A_{2l}^{(2)}$ or $D_{l+1}^{(2)}$ is said to be maximal if $\lambda + \delta$ is not a weight. It is known that the totality of the maximal weights of $L(\Lambda_0)$ is given by the Weyl group orbit through the highest weight Λ_0 , and hence each maximal weight is of multiplicity 1 [7]. Since the Weyl group can be seen as a subgroup of the group corresponding to the affine Lie algebra, the maximal weight vectors of $L(\Lambda_0)$ of $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$ are polynomial solutions to the $(2l+1)$ -reduced and $(2l+2)$ -reduced BKP hierarchy, respectively [3].

As we have shown previously, for each weight vector in $L(\Lambda_0)$ we can associate a symmetric state of the Maya game. We have also given a description, by means of the Maya game, how to obtain weight vectors of weight $\lambda \pm \delta$ for a given weight vector of weight λ . The weight λ is maximal if the corresponding weight vectors is at stalemate according to the rule.

Let us give a more precise description. For $A_{2l}^{(2)}$ the weight vector (m_1, \dots, m_{2d}) is a maximal weight vector if each m_j cannot be moved by $2l+1$ to the left and there are no such pairs $(m_i, m_j) = (k, 2l+1-k)$ for $1 \leq k \leq l$. For $D_{l+1}^{(2)}$, (m_1, \dots, m_{2d}) is a maximal weight vector if there are no m_j such that $m_j \equiv 0 \pmod{l+1}$ and there are no pairs (m_i, m_j) such that $m_i + m_j = n(2l+2)$ for $n \geq 1$. In particular, for the case $A_2^{(2)}$ the following set covers the maximal weight vectors:

$$\{\phi, (1, 4, 7, \dots, 3n-2), (2, 5, 8, \dots, 3n-1); n \geq 1\}.$$

For $D_4^{(2)}$ the maximal weight vectors of degree up to 12 are

$$\{\phi, (1), (2), (3), (1, 2), (1, 3), (5), (2, 3), (6), (1, 5), (1, 2, 3), (7), (1, 6), (2, 5), (1, 2, 5), (2, 7), (3, 6), (1, 9), (3, 7), (1, 3, 6), (5, 6), (2, 10), (5, 7), (1, 2, 9), (1, 5, 6), (2, 3, 7)\}.$$

If one uses the language of Young diagrams, the above criterion is stated in simpler words.

Theorem. Schur's Q -function $Q_Y(t')$ is a maximal weight vector of $L(\Lambda_0)$ of $A_{2l}^{(2)}$ (resp. $D_{l+1}^{(2)}$) if the shift symmetric diagram \tilde{Y} of Y has no hooks of length divisible by $2l+1$ (resp. $2l+2$).

In both cases, $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$, one recovers the (non-reduced) BKP hierarchy by taking the limit $l \rightarrow \infty$. Hence, as a corollary to the above theorem, we see that the Q -function $Q_Y(t)$ of any strict Young diagram Y solves the BKP hierarchy [9].

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